# Automorphisms $\phi$ of $\mathbb{Z}[x, y]$ and $f \in \mathbb{Z}[x, y]$ fixed by $\phi$

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## Motivating Question

Construct an algorithm which, for a given polynomial $f \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$, finds (explicitly in terms of generators) the maximal subgroup $G_f$ of the group $Aut(\mathbb{Z}[x_1, x_2, \ldots, x_n])$ that leaves $f$ fixed.[1]

## Automorphisms of $\mathbb{Z}[x]$

**Find the Automorphisms of $\mathbb{Z}[x]$**

Let $\phi$ be an automorphism on $\mathbb{Z}[x]$. The polynomial ring $\mathbb{Z}[x]$ is generated by $\{1, x\}$. It is known that automorphisms must map the identity to itself, so $\phi(1) = 1$.

We examine possible images of $x$. Invoking the surjectivity of $\phi$ restricts $\phi$ to degree 1 polynomials. Furthermore, invoking injectivity of $\phi$ restricts automorphisms to the form of

$$\phi : (1, x) \mapsto (1, \pm x + n).$$

**Automorphisms that fix $f \in \mathbb{Z}[x]$**

When asking which $\phi$ fix a given polynomial $f \in \mathbb{Z}[x]$, we see that shifting the polynomial by $n \neq 0$ will never fix the polynomial. For $\phi(x) = -x$, $\phi$ fixes even functions.

## Strategies for Finding Automorphisms

- Find explicit mappings for the *generators* of the polynomial ring. The automorphisms will be defined based upon these.
- Restrict the complexity of possible mappings by invoking properties of automorphisms.
- Study the automorphisms as a group under composition.
- We ignore constant shifts.

## Expanding to Automorphisms of $\mathbb{Z}[x, y]$

Let $\phi$ be an automorphism on $\mathbb{Z}[x, y]$. The polynomial ring $\mathbb{Z}[x, y]$ is generated by $\{1, x, y\}$. Both $\phi(x)$ and $\phi(y)$ could have the complexity of any degree polynomial in both $x$ and $y$. We are unable to leverage properties of automorphisms to restrict the degree of the automorphism.

Research in automorphisms over polynomial rings has classified *elementary* automorphisms.[2] An elementary automorphism over $\mathbb{Z}[x, y]$ is as follows

$$\phi : (1, x, y) \mapsto (1, \alpha x + f, y)$$

for some $f \in \mathbb{Z}[y]$. Likewise we can fix $x$.

It is known that the set of elementary automorphisms generates all elements in $Aut(\mathbb{Z}[x, y])$ under function composition.

## First Degree Automorphisms of $\mathbb{Z}[x, y]$

For now we restrict our research to degree one elementary automorphisms. We invoke the bijectivity constraint of automorphisms on this set and find that all possible degree 1 elementary automorphisms are of the form

$$\phi : (1, x, y) \mapsto (1, \pm x + By, y)$$
$$\text{or } \phi : (1, x, y) \mapsto (1, x, Cx \pm y).$$

We study the composition of these functions, which gives all degree one automorphisms on $\mathbb{Z}[x, y]$.

## References

[1] The Kourovka Notebook No. 19 arXiv:1401.0300v16 (2019)
[2] Ivan P. Shestakov, Ualbai U. Umirbaev, The Tame and the Wild Automorphisms of Polynomial Rings in Three Variables, J. Amer. Math. Soc. (2003)

## Group of Automorphisms

When examining all degree one automorphisms of the form

$$\phi_{ABCD} : (1, x, y) = (1, Ax + By, Cx + Dy)$$

under function composition, we find that the behavior of this group is isomorphic to $2 \times 2$ matrices under matrix multiplication. Formally we can say

$$\phi_{ABCD} \circ \phi_{EFGH} \cong \begin{bmatrix} A & C \\ B & D \end{bmatrix} \times \begin{bmatrix} E & G \\ F & H \end{bmatrix}.$$

The determinant of the elementary automorphisms in question must be 1 or $-1$. Then the composition of the elementary automorphisms also has determinant of 1 or $-1$. Formally we say

$$|\phi_{ABCD}| = AD - BC = \pm 1.$$

## Functions Fixed by Linear Automorphisms

Automorphisms of the form $\phi_{ABCD}$ where $AD - BC = 1$ fix

$$f = Cx^2 + (D - A)xy - By^2.$$

This single observation can be expanded to a family of functions using the properties of automorphisms. That is, $\phi$ fixes every sum of powers of $f$.

Exploration and analysis in SageMath did not reveal any automorphisms with $AD - BC = -1$ that fix any polynomials.

## Further Study

There is no intuition that says that the function fixed by $\phi_{ABCD}$ is the only one of its kind. If another function form were found, then the composition of both forms could be studied to find a group of functions that are fixed by automorphisms of $\mathbb{Z}[x, y]$. Classifying families of functions would allows us to know the structure of automorphisms that fix them.